

A Characterization of Tetrahedral Graphs*

R. C. BOSE AND RENU LASKAR

*Department of Statistics, University of North Carolina,
Chapel Hill, North Carolina 27514*

ABSTRACT

A tetrahedral graph may be defined as a graph G , whose vertices may be identified with the $n(n-1)(n-2)/6$ unordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two symbols in common. If $d(x, y)$ denotes the distance between two vertices x and y and $\Delta(x, y)$ denotes the number of vertices adjacent to both x and y , then a tetrahedral graph G has the following properties: (b₁) The number of vertices is $n(n-1)(n-2)/6$. (b₂) G is connected and regular of valence $3(n-3)$. (b₃) For any two adjacent vertices x and y , $\Delta(x, y) = n-2$. (b₄) $\Delta(x, y) = 4$ if $d(x, y) = 2$. We show that, if $n > 16$, then any graph G (without loops and with utmost one edge connecting two vertices) having the properties (b₁)—(b₄) must be a tetrahedral graph.

I. INTRODUCTION

1. We shall consider only finite undirected graphs, with at most one edge joining a pair of vertices and no edge joining a vertex to itself.

The valence $d(u)$ of the vertex u of a graph G , is defined to be the number of vertices adjacent to u . If all vertices of G have the same valence n_1 , the graph G is said to be a regular graph of valence n_1 .

A chain x_1, x_2, \dots, x_n is a sequence of vertices of G , not necessarily all different, such that any two consecutive vertices in the chain are adjacent. Thus the pairs $(x_1, x_2), (x_2, x_3), \dots, (x_{n-1}, x_n)$ are edges of G . The number of edges $n-1$ is said to be the length of the chain. The chain is said to begin at x_1 and terminate at x_n , and is said to join x_1 and x_n .

The graph G is said to be connected if, for every pair of distinct vertices x and y , there is a chain beginning at x and terminating at y . For a connected graph the distance $d(x, y)$ between two vertices x and y is defined to be the length of the shortest chain joining x and y .

For any two vertices u and v , $\Delta(u, v)$ denotes the number of vertices w ,

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adjacent to both u and v . If u and v are adjacent, i.e., $d(x, y) = 1$, $\Delta(u, v)$ is called the edge-degree of the edge (u, v) . A regular graph G for which all edges have the same edge-degree Δ , is said to be edge-regular, with edge-degree Δ .

2. A graph G is said to be triangular if the vertices of G can be identified with unordered pairs on n symbols, so that two pairs are adjacent if and only if the corresponding pairs have one symbol in common. A triangular graph G obviously possesses the following properties:

- (a₁) The number of vertices in G is $n(n-1)/2$.
- (a₂) G is regular of valence $2(n-2)$.
- (a₃) G is edge-regular with edge-degree $n-2$, i.e., $\Delta(u, v) = n-2$ if u and v are adjacent.
- (a₄) $\Delta(u, v) = 4$, if u and v are non-adjacent.

Connor [4] showed (with a somewhat different terminology) that, for $n > 8$, the properties (a₁)–(a₄), characterize a triangular graph, i.e., if G has the properties (a₁)–(a₄), then G must be triangular. Shrikhande [9], Li-chien [7, 8] and Hoffman [5, 6] completed Connor's work by demonstrating that the same result holds for $n < 8$, but if $n = 8$, then there exist other non-isomorphic graphs which are not triangular.

3. In this paper we consider the problem of characterization of tetrahedral graphs. A tetrahedral graph may be defined as a graph G whose vertices can be identified with unordered triplets on n symbols, such that two vertices are adjacent if and only if the corresponding triplets have two common symbols. It is readily seen that G has the following properties:

- (b₁) The number of vertices in G is $n(n-1)(n-2)/6$.
- (b₂) G is connected and regular of valence $3(n-3)$.
- (b₃) G is edge-regular, with edge degree $n-2$, i.e. $\Delta(x, y) = n-2$ if $d(x, y) = 1$.
- (b₄) $\Delta(x, y) = 4$ if $d(x, y) = 2$.

In Section III we prove that, for $n > 16$, the properties (b₁)–(b₄) characterize a tetrahedral graph, i.e., if G possesses properties (b₁)–(b₄) and $n > 16$, then it is possible to establish a (1,1) correspondence between the vertices of G , and the unordered triplets on n symbols, such that two vertices of G are adjacent if and only if the corresponding triplets have a pair of symbols in common.

The proof is based on certain theorems regarding the existence or non-existence of cliques and claws in edge-regular graphs, which are proved in Section II. These theorems generalize the previous work of Bruck [2] and (one of the authors) Bose [1]. Other applications of these theorems will be given in subsequent communications.

II. CLAWS AND CLIQUES IN EDGE-REGULAR GRAPHS

1. In this section we shall consider a graph G which has the following properties:

- (c₁) G is connected and regular of valence $r(k-1)$.
- (c₂) G is edge-regular with edge-degree $(k-2) + \alpha$.
- (c₃) $\Delta(x, y) \leq 1 + \beta$, for all pairs of non-adjacent vertices, x and y of G .

In the above r, k, α, β , are fixed positive integers, such that $r \geq 1$, $k \geq 2$, $\alpha \geq 0$, $\beta \geq 0$, and $r\beta - 2\alpha \geq 0$. All the lemmas and theorems in paragraphs 2 and 3 of this section are about the edge regular graph G , with properties (c₁), (c₂), (c₃).

We define here some functions of the parameters r, k, α , and β , which play an important role in subsequent developments.

- (2.1.1) $\gamma(r, \alpha) = 1 + (r-1)\alpha$.
- (2.1.2) $q(r, \alpha) = 1 + (2r-1)\alpha$.
- (2.1.3) $\rho(r, \alpha, \beta) = 1 + \beta + (2r-1)\alpha$.
- (2.1.4) $p(r, \alpha, \beta) = 1 + \frac{1}{2}(r+1)(r\beta - 2\alpha)$.

We shall denote as usual the cardinality of a set S by $|S|$.

A *clique* K of a graph is a set of vertices adjacent to each other. A clique K will be called *complete* if we cannot find a vertex x , not contained in K such that xUK is a clique. Thus a complete clique cannot be extended to a larger clique by the adjunction of a new vertex belonging to the graph.

Now consider the graph G with the properties, (c₁), (c₂), (c₃). A clique K of G will be called a *major clique* if

$$(2.1.5) \quad |K| \geq 1 + k - \gamma(r, \alpha) = k - (r-1)\alpha.$$

A clique K of G will be called a *grand clique* if it is both major and complete.

A claw $[p, S]$ of G , consists of a vertex p , the vertex of the claw, and a non-empty set S of vertices of G , not containing p , such that p is adjacent to every vertex in S , but any two vertices in S are non-adjacent. The order of the claw is defined to be the number $s = |S|$.

In the next two paragraphs of this section, we obtain a number of theorems about claws and cliques in a graph G having the properties (c₁), (c₂), (c₃). These theorems are very similar to those obtained in an earlier paper [1], but are now proved under less restrictive conditions, thereby substantially increasing their range of applicability. This will be illustrated in Section III, where they will be used to obtain a geometric characterization of tetrahedral graphs (defined in Section I). Further applications will be given in subsequent communications.

2. THEOREM (2.2.1). *If $k > p(r, \alpha, \beta)$, there cannot exist a claw of order $r + 1$ in G .*

Suppose there exists in G a claw $[p, S]$ of order s . Let T be the set of vertices of G , not belonging to $[p, S]$, and adjacent to p . Let $f(x)$ denote the number of vertices q in T , such that q is adjacent to exactly x vertices in S . Counting the number of vertices in T we have from (c₁),

$$(2.2.1) \quad \sum_{x=0}^s f(x) = r(k-1) - s = rk - r - s.$$

Counting the number of ordered pairs (b, q) where b and q are adjacent, b belongs to S , and q belongs to T , we have from (c₂)

$$(2.2.2) \quad \sum_{x=0}^s xf(x) = s(k-2+\alpha).$$

Again counting the triplets (b_1, b_2, q) , where b_1, b_2 is an ordered pair of vertices in S , q is a vertex in T , and b_1, b_2 are both adjacent to q , we have from (c₃)

$$(2.2.3) \quad \sum_{x=0}^s x(x-1)f(x) \leq s(s-1)\beta.$$

If a claw of order $r + 1$ exists, putting $s = r + 1$, we have from (2.2.1), (2.2.2) and (2.2.3):

$$(2.2.4) \quad \begin{aligned} f(0) + \frac{1}{2} \sum_{x=1}^{r+1} (x-1)(x-2)f(x) &\leq -k + 1 + \frac{1}{2}(r+1)(r\beta - 2\alpha) \\ &= -k + p(r, \alpha, \beta) \end{aligned}$$

Since the left-hand side is essentially non-negative whereas $k > p(r, \alpha, \beta)$ by hypothesis, we have a contradiction. This proves our theorem.

THEOREM (2.2.2). *If $k > \gamma(r, \alpha)$, then any claw of G of order $s < r$, can be extended to a claw of order r .*

Suppose there exists in G a claw $[p, S]$ of order s . From (2.2.1) and (2.2.2):

$$(2.2.5) \quad f(0) - \sum_{x=1}^s (x-1)f(x) = (k-1)(r-s) - \alpha s.$$

If $k > \gamma(r, \alpha) = 1 + (r - 1)\alpha$, and $s < r$,

$$f(0) > \alpha r(r - s - 1) \geq 0.$$

Hence $f(0) > 0$, which shows that there exists a vertex q in T , which is not adjacent to any vertex of S . If $S^* = S \cup q$ we can extend the claw $[p, S]$ to the claw $[p, S^*]$ of order $s + 1$. If $s + 1 < r$ we can continue the process till we arrive at a claw of order r .

THEOREM (2.2.3). *Given a claw $[p, S]$ of G of order $r - 1$ there exist at least $k - \gamma(r, \alpha)$ distinct vertices q of G such that $[p, SUq]$ is a claw of order r .*

Putting $s = r - 1$, in (2.2.5) we have

$$\begin{aligned} f(0) &\geq k - 1 - \alpha(r - 1) \\ &= k - \gamma(r, \alpha). \end{aligned}$$

Hence there exist at least $k - \gamma(r, \alpha)$ vertices q in T such that $[p, SUq]$ is a claw of order r .

3. LEMMA (2.3.1). *If $k > \gamma(r, \alpha)$ and if G has no claw of order $r + 1$, then any pair of adjacent vertices p and q is contained in at least one major clique.*

From Theorem (2.2.2) we can extend the claw $[p, q]$ to a claw $[p, S]$ of order r . Let b_1, b_2, \dots, b_r be vertices in S other than q . Let Ω be the set of vertices ω , which when adjoined to $S - q$ give a claw $[p, S^*]$ of order r , where $S^* = (S - q)U\omega$. Of course q is contained in Ω and from theorem (2.2.3)

$$|\Omega| \geq k - \gamma(r, \alpha).$$

The vertices in Ω are all adjacent to one another. If any two were not adjacent they could be added to b_1, b_2, \dots, b_r to give a claw of order $r + 1$. Let $K = pU\Omega$. Then K is a major clique since

$$|K| \geq 1 + k - \gamma(r, \alpha).$$

COROLLARY (2.3.1). *In Lemma (2.3.1), the hypothesis may be replaced by*

$$k > \max [\gamma(r, \alpha), p(r, \alpha, \beta)].$$

This follows at once from theorem (2.2.1).

COROLLARY (2.3.2). *When the conditions of Lemma (2.3.1), or Corollary (2.3.1) are satisfied, then any pair of adjacent vertices p and q , is contained in at least one grand clique.*

We can extend the major clique K by adding new vertices till it is complete and therefore a grand clique.

LEMMA (2.3.2). *If K and L are cliques of G , and $K \cup L$ is not a clique, then $|K \cap L| \leq 1 + \beta$.*

Since $K \cup L$ is not a clique, there exists in $K \cup L$ a pair of vertices c and d which are non-adjacent, such that c belongs to K and d to L . Any vertex belonging to $K \cap L$ must be adjacent to both c and d . Hence $\Delta(c, d) \geq |K \cap L|$. The lemma follows from (c₃).

LEMMA (2.3.3). *If K and L are cliques of G and $K \cap L$ contains at least two vertices a and b , then $|K \cup L| \leq k + \alpha$.*

Every vertex in $K \cup L$, other than a and b is adjacent to both a and b . Hence

$$\Delta(a, b) \geq |K \cup L| - 2.$$

It follows from (c₁), that $|K \cup L| \leq k + \alpha$.

LEMMA (2.3.4). *If K and L are cliques of G , $K \cup L$ is not a clique and $K \cap L$ contains at least two vertices, then*

$$|K| + |L| \leq 1 + k + \alpha + \beta.$$

This follows at once from Lemmas (2.3.2) and (2.3.3) by noting that

$$|K| + |L| = |K \cap L| + |K \cup L|.$$

THEOREM (2.3.1A). *If $k > \rho(r, \alpha, \beta)$ and G has no claw of order $r + 1$, then any pair of adjacent vertices p and q is contained in one and only one grand clique.*

The existence of at least one grand clique containing p and q follows at once from corollary (2.3.2) by noting that $\rho(r, \alpha, \beta) \geq \gamma(r, \alpha)$.

Suppose there exist at least two distinct grand cliques K and L both containing the adjacent vertices p and q . Since K and L are complete, $K \cup L$ is not a clique. Hence from Lemma (2.3.4)

$$|K| + |L| \leq 1 + k + \alpha + \beta.$$

But K and L are both major cliques. Hence

$$|K| + |L| \geq 2\{k - (r - 1)\alpha\},$$

which shows that

$$k \leq 1 + \beta + (2r - 1)\alpha = \rho(r, \alpha, \beta),$$

contrary to the hypothesis.

The previous theorem can be written in the following alternative form:

THEOREM (2.3.1B). *If $k > \max [\rho(r, \alpha, \beta) \rho(r, \alpha, \beta)]$, then any two adjacent vertices p and q of G are contained in exactly one grand clique.*

This follows at once from Theorems (2.2.1) and (2.3.1A).

THEOREM (2.3.2A). *If $k > q(r, \alpha)$, there exists no claw of order $r + 1$ in G , and every pair of adjacent vertices of G is contained in utmost one grand clique of G , then each vertex of G is contained in exactly r grand cliques.*

If we note that $q(r, \alpha) \geq \gamma(r, \alpha)$ it follows from Corollary (2.3.2) that any pair of adjacent vertices is contained in exactly one grand clique. Again from Theorem (2.2.2), p is the vertex of at least one claw of order r . Let $[p, S]$ be a claw of order r , where $S = \{b_1, b_2, \dots, b_r\}$. As in Theorem (2.2.1), let T be the set of vertices not belonging to S , which are adjacent to p .

Let H_j be the set consisting of p, b_j , and q belonging to T , such that q is adjacent to b_j but not adjacent to b_i , $i \neq j$. As in Theorem (2.2.1) let $f(x)$ denote the number of vertices in T which are adjacent to exactly x vertices in S . Then $f(0) = 0$, otherwise there would exist a claw of order $r + 1$. Putting $s = r$ in (2.2.1) and (2.2.2), we have

$$(2.3.1) \quad \sum_{x=1}^r f(x) = r(k - 2).$$

$$(2.3.2) \quad \sum_{x=1}^r xf(x) = r(k - 2 + \alpha).$$

Hence

$$\sum_{x=2}^r (x - 1)f(x) = r\alpha.$$

Since

$$\begin{aligned} -f(1) + \sum_{x=1}^r f(x) &= \sum_{x=2}^r f(x) \\ &\leq \sum_{x=2}^r (x-1)f(x) \\ &= r\alpha, \end{aligned}$$

it follows that

$$(2.3.3) \quad f(1) \geq r(k-2-\alpha).$$

Any two vertices of H_j are adjacent to one another, otherwise there would exist a claw of order $r+1$. Thus H_j is a clique.

Put $H_j^* = H_j - (b_j \cup p)$. Then H_j^* consists of exactly those vertices of T which are adjacent to b_j but to no other vertex of S . Hence $H_1^*, H_2^*, \dots, H_r^*$ are disjoint sets, and the total number of vertices in these sets is $f(1)$.

Now there is a unique grand clique K_j containing b_j and p . The number of vertices in K_j cannot be less than the number of vertices in H_j . If possible let $|K_j| < |H_j|$. Since K_j is a grand clique it follows that H_j is a major clique and contained in some grand clique K_j' . Since b_j and p are contained in K_j and K_j' , they must coincide. Hence K_j contains H_j , which contradicts $|K_j| < |H_j|$.

Now consider the r grand cliques, K_1, K_2, \dots, K_r . Then $K_1 - p, K_2 - p, \dots, K_r - p$ are disjoint. For if $K_i - p$ and $K_j - p$, $i \neq j$, have a common vertex q , then K_i and K_j would coincide, and would contain both b_i and b_j , which is impossible since b_i is not adjacent to b_j . Remembering (2.3.3), we have

$$\begin{aligned} (2.3.4) \quad \sum_{j=1}^r |K_j - p| &\geq \sum |H_j - p| \\ &= r + \sum_{j=1}^r |H_j^*| \\ &= r + f(1) \\ &\geq r(k-1-\alpha). \end{aligned}$$

If possible, suppose there is another grand clique K_{r+1} containing p . The vertices in $K_{r+1} - p$ must be disjoint from the vertices in $K_1 - p, K_2 - p, \dots, K_r - p$. Since K_{r+1} is a grand and therefore a major

clique, $|K_{r+1} - p| \geq k - 1 - (r - 1)\alpha$. But from (c₁), the number of vertices adjacent to p is exactly $r(k - 1)$. Hence from (2.3.4)

$$r(k - 1) \geq \sum_{j=1}^{r+1} |K_j - p| \geq r(k - 1) + k - 1 - (2r - 1)\alpha.$$

$$\therefore k \leq 1 + (2r - 1)\alpha = q(r, \alpha),$$

which is a contradiction. Thus p is contained in exactly r grand cliques.

THEOREM (2.3.2B). *If $k > \rho(r, \alpha, \beta)$, and there exists no claw of order $r + 1$ in G , then each vertex of G is contained in exactly r grand cliques.*

This follows at once from the previous theorem and Theorem (2.3.1A), remembering $\rho(r, \alpha, \beta) \geq q(r, \alpha)$.

THEOREM (2.3.2C). *If $k > \max [\rho(r, \alpha, \beta), p(r, \alpha, \beta)]$, then each vertex in G is contained in exactly r grand cliques.*

This follows from the previous theorem and Theorem (2.2.1).

III. CHARACTERIZATION OF TETRAHEDRAL GRAPHS

1. As mentioned earlier in the introduction, a tetrahedral graph G is a graph whose vertices can be identified with the $n(n - 1)(n - 2)/6$ unordered triplets on n symbols, such that any two vertices are adjacent if and only if the corresponding triplets have a pair of common symbols. Then G clearly possesses the properties (b₁)–(b₄) given in Section I, paragraph 3. We shall prove here that, if $n > 16$, the converse also holds.

In the following lemmas G is a graph satisfying the conditions (b₁)–(b₄), and such that $n > 16$.

If we set $\gamma = 3$, $K = n - 2$, $\alpha = 2$, $\beta = 3$, then the conditions (b₂), (b₃), (b₄) are the same as (c₁), (c₂), (c₃) of Section II. Also from (2.1.1), (2.1.3), and (2.1.4)

$$\gamma(r, \alpha) = 5, \rho(r, \alpha, \beta) = 14, p(r, \alpha, \beta) = 11.$$

Hence a clique K of G is a major clique if $|K| \geq n - 6$, and if it is complete it is a grand clique. Since $n > 16$, the condition $K > \max [\rho(r, \alpha, \beta), p(r, \alpha, \beta)]$ is satisfied. Hence from theorems (2.3.1B) and (2.3.2C) we have:

LEMMA (3.1.1). *Any two adjacent vertices of G are contained in exactly one grand clique. Each vertex of G is contained in exactly 3 grand cliques.*

The unique grand clique containing any two given adjacent vertices x and y , may be denoted by $K(x, y)$.

The null set will be denoted by \emptyset .

The following six lemmas are directed toward proving that $|K| = n - 2$, for any grand clique K in G .

LEMMA (3.1.2). If K is a grand clique in G , then

$$n - 4 \leq |K| \leq n.$$

Let x and y be any two vertices in K . There are $|K| - 2$ vertices in K other than x and y , and by definition each of these is adjacent to both x and y . If $|K| > n$, then $|K| - 2 > n - 2$, which would contradict (b_3) . Hence $|K| \leq n$.

Let A be the set of all vertices adjacent to both x and y but not contained in K . Then from (b_4)

$$|K| - 2 + |A| = n - 2.$$

i.e.,

$$(3.1.1) \quad |K| + |A| = n.$$

If S_1 and S_2 are the two other cliques containing x and T_1, T_2 are those containing y , then any vertex in A must be of the form

$$Z_{ij} = S_i \cap T_j, i, j = 1, 2,$$

if it exists at all. Since two distinct cliques can have at most one vertex in common, we have

$$|A| \leq 4.$$

Hence from (3.1.1)

$$|K| \geq n - 4.$$

LEMMA (3.1.3). *If K is a grand clique in G , then*

$$|K| \neq n - 4.$$

Suppose $|K| = n - 4$, and let $x, y, A, S_i, T_j (i, j = 1, 2)$ be as in Lemma (3.1.2). Since there are only $n - 6$ vertices in K adjacent to both x

and y , we must have $|A| = 4$. Hence, $S_i \cap T_j \neq \emptyset$ for $i, j = 1, 2$. It follows from (b_2) that

$$(3.1.2) \quad |K - x| + |S_1 - x| + |S_2 - x| = 3n - 9.$$

Since $|K - x| = n - 5$, we have

$$(3.1.3) \quad |S_1 - x| + |S_2 - x| = 2n - 4.$$

Thus, at least one $S_i - x$, say $S_1 - x$, has at least $n - 2$ vertices, and therefore

$$(3.1.4) \quad |S_1| \geq n - 1.$$

Consider the two vertices x and z_{11} in S_1 . They are both adjacent to $|S_1| - 2$ other vertices in S_1 as well as to y and z_{21} not in S_1 . From (3.1.4) it follows that $\Delta(x, z_{11}) \geq n - 1$, which contradicts (b_3) .

LEMMA (3.1.4). *If K is a grand clique in G , then $|K| \neq n - 3$.*

Suppose $|K| = n - 3$. Then, from (3.1.1), $|A| = 3$, and one of the grand cliques, say S_1 , containing x must intersect both of the other two grand cliques containing y and the other grand clique S_2 , containing x , must intersect exactly one of T_1, T_2 , say T_2 .

Since $|K - x| = n - 4$, from (3.1.2) we have

$$(3.1.5) \quad |S_1 - x| + |S_2 - x| = 2n - 5.$$

It follows, then, that one of $S_1 - x$ and $S_2 - x$ has at least $n - 2$ vertices. If $|S_1 - x| \geq n - 2$, then $|S_1| \geq n - 1$ and considering the vertices x and z_{12} we have at most one vertex not in S_1 adjacent to both. This is contradicted, since z_{22} and y are adjacent to both. If $|S_2| \geq n - 1$, the same argument can be applied to x and z_{22} .

LEMMA (3.1.5). *If K is a grand clique in G , then $|K| \neq n - 1$.*

Suppose $|K| = n - 1$, then, from (3.1.1), $|A| = 1$. Hence, exactly one of S_1, S_2 intersects exactly one of T_1, T_2 .

Suppose $z_{11} = S_1 \cap T_1$ and $S_2 \cap T_2 = \emptyset$.

Since $|K - x| = n - 2$, it follows from (3.1.2) that

$$|S_1 - x| + |S_2 - x| = 2n - 7.$$

Hence for one i , $|S_i - x| \leq n - 4$ and then $|S_i| \leq n - 3$. But, by Lemmas (3.1.2), (3.1.3), (3.1.4), we have $|K| \geq n - 2$ for every K in G .

LEMMA (3.1.6). *If K is a grand clique in G , then*

$$|K| \neq n.$$

Suppose $|K| = n$, then $|A| = 0$ and hence $S_i \cap T_j = \emptyset$, ($i = 1, 2$), ($j = 1, 2$). Since $|K - x| = n - 1$, from (3.1.2) we have

$$|S_1 - x| + |S_2 - x| = 2n - 8.$$

Hence at least one of $S_1 - x$ and $S_2 - x$, say $S_1 - x$, has at most $n - 4$ vertices. Thus $|S_1| \leq n - 3$, which contradicts at least one of the Lemmas (3.1.2), (3.1.3), (3.1.4).

LEMMA (3.1.7). *If K is a grand clique in G , then $|K| = n - 2$.*

This follows immediately from Lemmas (3.1.2)–(3.1.6).

LEMMA (3.1.8). *Let x be a vertex in G and let L be a grand clique not containing x . Then the three grand cliques K_1, K_2, K_3 containing x cannot all intersect L .*

Suppose K_1, K_2, K_3 all meet L , and let $y_i = K_i \cap L$, $i = 1, 2, 3$. From Lemma (3.1.1) the vertices y_i , $i = 1, 2, 3$ are all distinct. Let S_i , $i = 1, 2, 3$ be the third grand clique containing y_i in addition to K_i and L .

Suppose $S_i \cap K_j \neq \emptyset$, for some pair i, j , $i \neq j$, and let $z = S_i \cap K_j$. Then, the vertices z , y_j , and y_k are such that, each of these is adjacent to both x and y_i but none contained in the grand clique K_i containing x and y_i (i, j, k are different). From Lemma (3.1.7) $|K_i| = n - 2$, and hence the $n - 4$ vertices in K_i , other than x and y_i , together with the three vertices z , y_j , and y_k , constitute a set of $n - 1$ vertices which are adjacent to both x and y_i . This contradicts (b_3) . Hence

$$(3.1.6) \quad S_i \cap K_j = \emptyset \text{ for } i \neq j, i, j = 1, 2, 3.$$

By Lemma (3.1.7) there are $n - 5$ vertices in L other than y_1, y_2 , and y_3 . Each of these vertices must be non-adjacent to x , for otherwise we would have 4 grand cliques containing x contradicting Lemma (3.1.1). Let z be one of these $n - 5$ vertices in $L - y_1 - y_2 - y_3$. Then, $d(z, x) = 2$ and by (b_4) there are exactly 4 vertices adjacent to both z and x . Clearly three of these are y_1, y_2, y_3 and the fourth vertex must be on some K_i ($1 \leq i \leq 3$) and be distinct from x and y_i . Thus for each of the $n - 5$ vertices in $L - y_1 - y_2 - y_3$, there is exactly one vertex in the set

$$T = \bigcup_{i=1}^3 \{K_i - x - y_i\}$$

which is adjacent to it. Let us define three sets A_1, A_2, A_3 where A_i consists of all those vertices in $K_i - x - y_i$ which are adjacent to at least one vertex of L other than y_i . Since from (3.1.6) no vertex $z' \in K_i - x - y_i$ can be adjacent to $y_j, j \neq i$, it is clear that A_i consists of all those vertices in $K_i - x - y_i$ which are adjacent to a vertex of $L - y_1 - y_2 - y_3$.

Now, since each vertex in $L - y_1 - y_2 - y_3$ is adjacent to exactly one vertex T , and since

$$|L - y_1 - y_2 - y_3| = n - 5,$$

we have

$$(3.1.7) \quad \sum_{i=1}^3 |A_i| \leq n - 5.$$

Again, since $|K_i - x - y_i| = n - 4$, and the $K_i - x - y_i, i = 1, 2, 3$ are disjoint, we have

$$(3.1.8) \quad |T| = \sum_{i=1}^3 |K_i - x - y_i| = 3n - 12.$$

If we define B_i to be the set of all those vertices in $K_i - x - y_i$ which are adjacent to no vertex of L other than y_i , then clearly A_i is disjoint from B_i and

$$A_i \cup B_i = K_i - x - y_i.$$

Hence, from (3.1.8),

$$(3.1.9) \quad \sum_{i=1}^3 |A_i| + \sum_{i=1}^3 |B_i| = (3n - 12).$$

Combining (3.1.7) and (3.1.9), we have

$$(3.1.10) \quad \sum_{i=1}^3 |B_i| \geq 2n - 7.$$

It follows from (3.1.10) that, for at least one $i = i_0$, we have

$$(3.1.11) \quad |B_{i_0}| \geq \frac{2n - 7}{3}.$$

Now, let $b \in B_{i_0}$ and consider the vertices b and y_{i_0} . Since $|K_{i_0}| = n - 2$ from Lemma (3.1.7), it follows from (b_3) that there are exactly two vertices adjacent to both b and y_{i_0} and not in K_{i_0} . From the definition of

B_{i_0} , neither of these two vertices can be in L . Hence they must both lie in $S_{i_0} - y_{i_0}$.

Let b and b' be two distinct vertices in B_{i_0} and let s_1, s_2 be the two vertices of $S_{i_0} - y_{i_0}$ adjacent to b and s'_1, s'_2 be the two vertices of $S_{i_0} - y_{i_0}$ adjacent to b' . Suppose

$$s'_i = s_j, \quad \text{for some pair } i, j, \quad 1 \leq i, j \leq 2.$$

Without loss of generality, suppose $s'_1 = s_1$. Let M be the grand clique containing the adjacent vertices b and s_1 . Then, there are $n - 4$ vertices in M adjacent to both b and s_1 as well as 3 others not in M , namely, s_2, b' , and y_{i_0} . It follows that there are at least $n - 1$ vertices adjacent to both b and s_1 which contradicts (b_3) . Hence the 4 vertices s_1, s_2, s'_1, s'_2 must all be distinct. Hence the number of distinct vertices in $S_{i_0} - y_{i_0}$ adjacent to vertices in B_{i_0} is exactly $2 |B_{i_0}|$. Consequently

$$(3.1.12) \quad |S_{i_0} - y_{i_0}| \geq 2 |B_{i_0}|$$

or, from (3.1.11) and Lemma (3.1.7),

$$n - 3 \geq 2 \left(\frac{2n - 7}{3} \right),$$

which contradicts the assumption $n > 16$. This completes the proof.

LEMMA (3.1.9). *Let x and y be two adjacent vertices in G and let K be the grand clique containing both x and y . Let S_1, S_2 be the other two grand cliques containing x and T_1, T_2 be the other two grand cliques containing y . Then the grand cliques S_i may be put in (1,1) correspondence with grand cliques T_j so that only corresponding cliques intersect,*

$$S_i \cap T_i \neq \emptyset,$$

$$S_i \cap T_j = \emptyset, \quad i \neq j.$$

Since $|K| = n - 2$, it is clear that exactly two of the 4 intersections

$$S_i \cap T_j, \quad i, j = 1, 2$$

are non-empty. If S_1 and S_2 both intersect one of the grand cliques containing y , other than K , say T_1 , then all three grand cliques containing x intersect T_1 and $x \notin T_1$. This contradicts Lemma (3.1.8). Hence each of the grand cliques S_i intersects one and only one of the grand cliques and vice versa.

LEMMA (3.1.10). *Let x and y be two vertices in G such that, $d(x, y) = 2$. Then there is one grand clique S_3 containing x which does not intersect any grand clique containing y , and one grand clique T_3 containing y which does not intersect any grand clique containing x . The other two grand cliques S_1, S_2 containing x and the other two grand cliques T_1, T_2 containing y mutually intersect.*

From (b_4) , there are exactly 4 vertices adjacent to both x and y . Let z_{11} be one of these and let S_1 be the grand clique containing x and z_{11} and T_1 be the grand clique containing y and z_{11} . Clearly $S_1 \neq T_1$, since $d(x, y) = 2$.

Since $d(x, z_{11}) = 1$ by Lemma (3.1.9) there exists exactly one grand clique S_2 containing x , other than S_1 , which intersects T_1 . Let $z_{21} = S_2 \cap T_1$. Clearly $z_{11} \neq z_{21}$, $y \neq z_{21}$. Similarly since $d(z_{11}, y) = 1$, there exists exactly one grand clique T_2 containing y , which intersects S_1 in z_{12} say. Clearly $x \neq z_{12}$, $z_{11} \neq z_{12}$. Suppose $T_2 \cap S_2 = \emptyset$. Then since $d(z_{12}, x) = 1$, from the previous lemma the third grand clique S_3 containing x must intersect T_2 in z_{32} , say. Similarly, the third grand clique T_3 containing y must intersect S_2 in z_{23} , say. Since $z_{23} \in S_2$ and $z_{32} \in T_2$ and $S_2 \cap T_2 = \emptyset$, we must have $z_{23} \neq z_{32}$. But then we have 5 vertices adjacent to both x and y , namely, $z_{11}, z_{12}, z_{21}, z_{23}, z_{32}$. This contradicts (b_4) and hence $S_2 \cap T_2 \neq \emptyset$. Let $z_{22} = S_2 \cap T_2$, we then have 4 vertices adjacent to both x and y , namely, z_{ij} , $i, j = 1, 2$. It follows from (b_4) that

$$S_3 \cap T_j = \emptyset, \quad j = 1, 2, 3,$$

$$S_i \cap T_3 = \emptyset, \quad i = 1, 2, 3.$$

LEMMA (3.1.11). *Given two distinct grand cliques K_1 and K_2 of G , with a common vertex x , there is a $(1,1)$ correspondence between the vertices of K_1 and K_2 such that the corresponding vertices are contained in a grand clique.*

Let y_1 be a vertex of K_1 , $y_1 \neq x$. If $z \neq x$ is a vertex of K_2 , then $d(y_1, z) \leq 2$. It follows as in the previous lemma that every vertex of K_2 is not adjacent to y_1 . Hence there exists a vertex z in K_2 such that $d(y_1, z) = 2$. From Lemma (3.1.10) there is another clique K^* , besides K_1 , which contains y_1 and intersects K_2 . Let $y_2 = K^* \cap K_2$. Then y_1 is adjacent to y_2 . Also the third clique containing y_1 does not intersect K_2 . Thus there is exactly one vertex y_2 in K_2 which is adjacent to y_1 . In the same way we can start from a vertex y_2 in K_2 and show that there is exactly one vertex in K_1 adjacent to y_1 .

LEMMA (3.1.12). *There are exactly $n(n-1)/2$ grand cliques in G .*

Consider ordered pairs (x, K) where x is a vertex and K is a grand clique of G containing x . Since each vertex is contained in 3 grand cliques we get $3v$ such pairs, where $v = n(n-1)(n-2)/6$ is the number of vertices in G . But each grand clique accounts for $n-2$ pairs. Hence the number of grand cliques is $3v/(n-2) = n(n-1)/2$.

LEMMA (3.1.13). *Each grand clique in G is intersected by exactly $2(n-2)$ other grand cliques.*

This follows at once by noting that each vertex of a grand clique K is contained in exactly two other grand cliques.

LEMMA (3.1.14). *If K_1 and K_2 are two intersecting grand cliques in G , there exist exactly $n-2$ grand cliques which intersect both K_1 and K_2 .*

Let $x = K_1 \cap K_2$. Then through x there passes another grand clique K_3 (intersecting K_1 and K_2 in x). Again if y_1 is any one of the other $n-3$ points on K_1 , then from Lemma (3.1.11) there exists a corresponding point y_2 on K_2 such that y_1 and y_2 are adjacent, $K(y_1, y_2)$ being a grand clique intersecting both K_1 and K_2 . Each pair of corresponding points gives one such clique and vice versa.

The following three lemmas are directed toward proving that there exist exactly 4 grand cliques, which intersect each of two given non-intersecting grand cliques K_1 and K_2 .

LEMMA (3.1.15). *If K_1 and K_2 are two non-intersecting grand cliques in G , x_1 and x_2 are two adjacent vertices belonging to K_1 and K_2 , respectively, then there exist vertices y_1, y_2 belonging to K_1 and K_2 , respectively, such that x_1, x_2, y_1, y_2 are mutually adjacent ($y_1 \neq x_1, y_2 \neq x_2$).*

Since K_1 and $K(x_1, x_2)$ are intersecting cliques, by Lemma (3.1.11), there exists a vertex y_1 in K_1 which corresponds to x_2 and is therefore adjacent to it. Similarly there exists a vertex y_2 in K_2 , which is adjacent to x_1 .

Clearly $d(y_1, y_2) \leq 2$. We want to show that y_1 and y_2 are adjacent. If not, then $d(y_1, y_2) = 2$. Then by Lemma (3.1.10) there exists a clique K_1^* containing y_1 , and a clique K_2^* containing y_2 such that K_1^* does not intersect any clique containing y_2 , K_2^* does not intersect any clique containing y_1 . Hence K_1^* must be distinct from K_1 and $K(y_1, x_2)$. Also K_2^* must be distinct from K_2 and $K(y_2, x_1)$. The cliques K_1 and $K(y_1, x_2)$ must therefore intersect the cliques K_2 and $K(y_2, x_1)$. Since by hypothesis K_1 does not intersect K_2 , this is a contradiction. Hence $d(y_1, y_2) = 1$.

LEMMA (3.1.16). *If K_1 and K_2 are two non-intersecting grand cliques in G , such that there exist vertices x_1 and x_2 belonging to K_1 and K_2 , respectively, with $d(x_1, x_2) = 2$, then there exists at least one pair of adjacent vertices one of which belongs to K_1 and the other to K_2 .*

From Lemma (3.1.10), there exist two cliques (different from K_1) containing x_1 , which intersect each of the two cliques (other than K_2) containing x_2 . Let z be the vertex of intersection of one of the cliques containing x_1 with one of the cliques containing x_2 . Then z is different from x_1 and x_2 . Since K_1 and $K(x_1, z)$ are intersecting cliques, from Lemma (3.1.11), there exists a vertex y_1 in K_1 ($y_1 \neq x_1$), such that y_1 is adjacent to z . Similarly there exists a vertex y_2 in K_2 adjacent to z . x_1 and x_2 are non-adjacent by hypothesis. If the lemma is false then the pairs (x_1, y_2) , (y_1, x_2) , and (y_1, y_2) must also be non-adjacent. Hence $K(z, x_1)$, $K(z, x_2)$, $K(z, y_1)$, and $K(z, y_2)$ must all be different grand cliques, contradicting the Lemma (3.1.1).

LEMMA (3.1.17). *If K_1 and K_2 are two non-intersecting grand cliques in G , there must exist a pair of adjacent vertices one of which belongs to K_1 and the other to K_2 .*

Let x_1 be a vertex in K_1 and x_2 a vertex in K_2 . If $d(x_1, x_2) = 1$, the lemma is true. If $d(x_1, x_2) = 2$, the required result follows from the previous lemma. Suppose $d(x_1, x_2) = r > 2$. Then, there exists a chain $x_1, z_1, z_2, \dots, z_{r-1}, x_2$ joining x_1 and x_2 . Clearly $d(z_{r-2}, x_2) = 2$. Consider the grand cliques $K(z_{r-3}, z_{r-2})$ and K_2 . These must be non-intersecting, otherwise $d(z_{r-3}, x_2) = 2$, and x and y would be at distance $r - 1$. Hence from the previous lemma we can find a vertex y_2 in $K(z_{r-3}, z_{r-2})$ and z_{r-2}^* in K_2 such that y_2 and z_{r-2}^* are adjacent. Then $d(x_1, y_2) = r - 1$. By successive repetition of the same process we can find a vertex w_2 in K_2 such that $d(x_1, w_2) = 2$. The required result then follows from the previous lemma.

LEMMA (3.1.18). *If K_1 and K_2 are two non-intersecting grand cliques in G , then there exist exactly 4 grand cliques, which intersect each of K_1 and K_2 .*

From the previous lemma we can find a vertex x_1 in K_1 and a vertex x_2 in K_2 such that x_1 is adjacent to x_2 . Now from Lemma (3.1.15) we can find in K_1 a vertex $y_1 \neq x_1$, and in K_2 a vertex $y_2 \neq x_2$, such that x_1, x_2, y_1, y_2 are mutually adjacent. Hence $K(x_1, x_2)$, $K(x_1, y_2)$, $K(y_1, x_2)$, and $K(y_1, y_2)$ are 4 grand cliques each of which intersects K_1 and K_2 .

If possible let there be another grand clique which intersects K_1 in z_1 and K_2 in z_2 . Then z_1 is distinct from x_1 and y_1 and z_2 is distinct from x_2 and y_2 .

Otherwise there would be 4 grand cliques containing some vertex. Again z_1 is non-adjacent to x_2 . Otherwise there would be 4 grand cliques containing x_2 . Hence $d(z_1, x_2) = 2$. The grand cliques K_1 and K_2 containing z_1 and x_2 , respectively, are non-intersecting. Since the grand cliques $K(x_2, x_1)$ and $K(x_2, y_1)$ intersect the grand clique K_1 containing z_1 , it follows from Lemma (3.1.10), that K_2 , the third grand clique containing x_2 , cannot intersect any grand clique containing z_1 . This is a contradiction, since $K(z_1, z_2)$ intersects K_2 in z_2 .

2. THEOREM (3.2.1). *For a tetrahedral graph G , the conditions (b_1) – (b_4) hold. Conversely if $n > 16$ and conditions (b_1) – (b_4) are satisfied then G is tetrahedral.*

Given n symbols, $1, 2, \dots, n$, a tetrahedral graph G is the graph whose vertices are the unordered triplets on these symbols, two triplets being adjacent when the corresponding triplets have two symbols in common. The conditions (b_1) – (b_4) are easily checked.

(i) The number of unordered triplets on n symbols is clearly $n(n-1)(n-2)/6$, which is the number of vertices in G .

(ii) Let x be the vertex corresponding to the symbol (i, j, k) , for a vertex adjacent to x , the corresponding triplet must contain 2 of the symbols i, j, k and one of $n-3$ symbols other than i, j, k . Hence the number of vertices adjacent to x is $3(n-3)$. Hence G is regular of valence $3(n-3)$.

Given any two triplets, we can readily construct a chain of triplets beginning with the first and ending with the second so that two consecutive triplets have two symbols in common. Hence G is connected.

(iii) Let x and y be two adjacent vertices of G . Let x and y correspond to (i, j, k_1) and (i, j, k_2) . Then the triplets which have two symbols in common with each of the two triplets are the triplets (i, k_1, k_2) , (j, k_1, k_2) and the $n-4$ triplets (i, j, s) , where s is any of the n symbols other than i, j, k_1, k_2 . Hence the number of vertices adjacent to both x and y is $n-2$, i.e., $\Delta(x, y) = n-2$.

(iv) Let x and y be two vertices such that $d(x, y) = 2$. Then we may take x to correspond to (i, j_1, k_1) and y to correspond to (i, j_2, k_2) . Then the only triplets which have two symbols in common with both these triplets are (i, j_1, j_2) , (i, j_1, k_2) , (i, j_2, k_1) , and (i, k_1, k_2) . Thus $\Delta(x, y) = 4$ if $d(x, y) = 2$.

Conversely, suppose $n > 16$ and the conditions (b_1) – (b_4) are satisfied for G . Then Lemmas (3.1.1)–(3.1.18) hold. Let H be a graph whose vertices are the grand cliques of G , two vertices of H being adjacent if and

only if the corresponding grand cliques of G have a non-empty intersection. Then H satisfies the following conditions:

- (a₁) The number of vertices in H is $n(n - 1)/2$.
- (a₂) H is regular and of valence $2(n - 2)$.
- (a₃) H is edge regular with edge degree $n - 2$, i.e., $\Delta(u, v) = n - 2$ if u and v are adjacent.
- (a₄) $\Delta(u, v) = 4$ if u and v are non-adjacent.

Conditions (a₁), (a₂), (a₃), (a₄) are satisfied in virtue of Lemmas (3.1.12), (3.1.13), (3.1.14), and (3.1.18). It now follows from Connor's theorem (see Section I) that, for $n > 8$, H is triangular. Hence to each vertex of H we can associate an unordered pair (i, j) of two distinct symbols chosen out of n symbols $1, 2, \dots, n$ so that two vertices of H are adjacent if and only if the corresponding pair have a common symbol. From the correspondence between H and G it follows that to each grand clique in G we can associate an unordered pair of symbols chosen out of n symbols such that the pairs corresponding to two grand cliques have a common symbol if and only if the two grand cliques intersect.

Let K_1 and K_2 be any two intersecting grand cliques of G , having the vertex x in common, and let (i, j) and (i, k) be the corresponding pairs. To the vertex x we associate the unordered triplet (i, j, k) . The third grand clique K_3 containing x , intersects both K_1 and K_2 and hence must correspond to the pair (j, k) . Note that the triplet assigned to x is unambiguously determined by any two of the three cliques intersecting in x . Thus to each of the $n(n - 1)(n - 2)/6$ vertices of G we can associate a unique triplet.

If two vertices x and y of G are adjacent then there is a unique grand clique K containing x and y . If (i, j) is the pair corresponding to K , then the triplets corresponding to x and y must contain the symbols i and j . Conversely if the triplets corresponding to the vertices x and y contain the symbols i and j , then x and y are contained in the grand clique K corresponding to the pair (i, j) , and must therefore be adjacent. Thus two vertices of G are adjacent if and only if the corresponding triplets have a common pair of symbols. Hence G must be tetrahedral.

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